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On Bäcklund transformations for many-body systems

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Abstract. Using the n-particle periodic Toda lattice and the relativistic generalization due to Ruijsenaars of the elliptic Calogero–Moser system as examples, we revise the basic properties of the Bäcklund transformations (BTs) from the Hamiltonian point of view. The analogy between the BT and Baxter's quantum Q-operator pointed out by Pasquier and Gaudin is exploited to produce a conjugated variable μ for the parameter λ of the BT B_λ, such that μ belongs to the spectrum of the Lax operator $L(\lambda)$. As a consequence, the generating function of the composition $B_{\lambda_1} \circ \ldots \circ B_{\lambda_n}$ of n BTs gives rise to another canonical transformation separating variables for the model. For the Toda lattice the dual BT parametrized by μ is introduced.

1. Introduction

Bäcklund transformations (BTs) are an important tool in the theory of integrable systems [1]. Most frequently, they are understood as special mappings between solutions of nonlinear evolution equations. The Hamiltonian properties of BTs, as canonical transformations, are less well studied. Recent developments in quantum integrable theories [2, 3], discrete-time dynamics [4–6] and separation of variables (SoV) [7, 8] suggest, however, that the Hamiltonian aspect of BTs deserves more attention.

The aim of the present paper is to revise the concept of BTs from the Hamiltonian point of view and to point out some new properties of BTs. We restrict our attention to the finite-dimensional integrable systems and illustrate our general remarks by the example of the periodic Toda lattice and the elliptic Ruijsenaars model. When elaborating our approach to BTs, we have benefited greatly from the works of Pasquier and Gaudin [2], where a fundamental relationship between the BT and Baxter's quantum Q-operator was discovered, and of Veselov [4], who gave us the adequate mathematical language to speak about integrable mappings.

In section 2 we remind ourselves of the main properties of Bäcklund transformations for Liouville integrable systems, and a new property of *spectrality* is introduced. The meaning of spectrality is elucidated by making the comparison with the Baxter's quantum Q-operator. It is shown that spectrality of the BT provides an effective solution to the problem of separation of variables. In two subsequent sections we illustrate the new property of BTs for two families of integrable many-body systems. In conclusion, section 5 contains a summary and a discussion.

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2. Spectrality and separation of variables

Suppose an integrable system with n degrees of freedom is described in terms of the canonical Darboux variables $X \equiv \{X_i\}_{i=1}^n$ and $x \equiv \{x_i\}_{i=1}^n$, with the Poisson brackets

$$\{X_i, X_j\} = \{x_i, x_j\} = 0 \qquad \{X_i, x_j\} = \delta_{ij} \tag{2.1}$$

and functionally independent commuting Hamiltonians $H_i \equiv H_i(X, x)$

$$\{H_i, H_j\} = 0 \qquad i, j = 1, \ldots, n. \tag{2.2}$$

For our purposes it is convenient to think of a BT as a canonical transformation B_λ from the canonical variables (X, x) to the canonical variables (Y, y). It is important that B_λ depends on a complex parameter λ. We shall suppose that B_λ can be described via the generating function $F_\lambda(y; x)$ such that

$$X_i = \frac{\partial F_\lambda}{\partial x_i} \qquad Y_i = -\frac{\partial F_\lambda}{\partial y_i}. \tag{2.3}$$

The list of properties defining a BT usually includes:

(i) canonicity, see earlier;

(ii) invariance of Hamiltonians

$$H_i(X, x) = H_i(Y, y) \qquad i = 1, \ldots, n; \tag{2.4}$$

(iii) commutativity

$$B_{\lambda_1} \circ B_{\lambda_2} = B_{\lambda_2} \circ B_{\lambda_1} \tag{2.5}$$

where $\circ$ means the composition of canonical transformations.

In the case of *algebraically integrable* systems [9] one more property can be added to the list:

(iv) Algebraicity. Equations (2.3) describing B_λ are supposed to be algebraic with respect to X, Y and properly chosen functions of x and y (say, exponential or elliptic).

In the present paper, however, we concentrate on the analytic properties of BTs and ignore their algebraic and algebro-geometric aspects.

It is important to make a clear distinction between the notion of a BT and the close notions of *integrable canonical mapping* [4], or *integrable discrete-time dynamics*. The latter two are defined by the properties of canonicity and invariance only, the parameter λ being disregarded. The term 'discrete-time dynamics' refers usually to the case when the canonical transformation degenerates, in a certain limit, into an infinitesimal generator $\{H, \cdot\}$ of a continuous Hamiltonian flow. Existence of the parameter λ is crucial for our definition of a BT and enriches it with new properties.

Although the commutativity of BTs is traditionally proved as an independent property, in fact it follows from the canonicity and the invariance of Hamiltonians. Indeed, as shown in [4], any integrable canonical mapping acts on the Liouville torus as a shift (or a collection of shifts, in the case of multivalued mappings) of the angle variables $\varphi_i \to \varphi_i + b_i(\lambda)$. The commutativity is then obvious.

The theory of BTs acquires a new aspect if the integrable system in question is solvable via the inverse scattering (or inverse spectral transform) method. Suppose that the commuting Hamiltonians H_i can be obtained as the coefficients of the characteristic polynomial

$$W(u, v; \{H_i\}) = \det(v - L(u)) \tag{2.6}$$

of a matrix $L(u) \equiv L(u; X, x)$ (Lax operator) depending on X, x and a complex parameter u. Note that the invariance of H_i under B_λ is equivalent then to the invariance of the spectrum of $L(u)$, that is there exists an invertible matrix $M(u)$ such that

$$M(u)L(u; X, x) = L(u; Y, y)M(u) \qquad \forall u \in \mathbb{C}. \tag{2.7}$$

The properties of BTs listed above are well known. Now we are going to add to the list a new property which is the main contribution of the present paper.

(v) Spectrality. Let μ be defined as the variable conjugated to λ

$$\mu = -\frac{\partial F_\lambda}{\partial \lambda}. \tag{2.8}$$

We shall say that the BT B_λ is *associated* to the Lax operator $L(u)$ if for some function $f(\mu)$ the pair $(\lambda, f(\mu))$ lies on the *spectral curve* of the Lax matrix

$$W(\lambda, f(\mu); \{H_i\}) \equiv \det(f(\mu) - L(\lambda)) = 0. \tag{2.9}$$

This *spectrality* property of BTs seems to be new, at least we failed to find it in the literature. We have verified it for the Toda lattice and the elliptic Ruijsenaars model for which $f(\mu) = \mathrm{e}^{-\mu}$ (see sections 3 and 4). It seems plausible, however, that spectrality is the property shared by BTs for a much larger class of models.

The meaning of the equality (2.9) becomes clear if we turn to the quantum case. In the pioneering paper by Pasquier and Gaudin [2], based on the earlier treatment of the classical Toda lattice by Gaudin [10], a remarkable connection was established between the classical BT B_λ for the Toda lattice and the famous Baxter's Q-operator [11]. Pasquier and Gaudin constructed a certain integral operator $\hat{Q}_\lambda$

$$\hat{Q}_\lambda : \Psi(x) \rightarrow \int \mathrm{d}x\, Q_\lambda(y; x)\Psi(x) \tag{2.10}$$

(here and below $\mathrm{d}x \equiv \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ etc) whose properties parallel those of the classical BT B_λ. In the quantum case the canonical transformation is replaced with the similarity transformation

$$\hat{Y}_i = \hat{Q}_\lambda \hat{X}_i \hat{Q}_\lambda^{-1} \qquad \hat{y}_i = \hat{Q}_\lambda \hat{x}_i \hat{Q}_\lambda^{-1} \tag{2.11}$$

where the hat ˆ distinguishes the quantum operators from their classical counterparts. The correspondence between the kernel $Q_\lambda(y; x)$ of $\hat{Q}_\lambda$ and the generating function $F_\lambda(y; x)$ of B_λ is given by the semiclassical relation

$$Q_\lambda(y; x) \sim \exp\left(-\frac{\mathrm{i}}{\hbar} F_\lambda(y; x)\right) \qquad \hbar \rightarrow 0. \tag{2.12}$$

After publication of [2] the Q-operators have been found for a number of other quantum integrable models [3].

The properties of $\hat{Q}_\lambda$ such as the invariance of the Hamiltonians

$$[\hat{Q}_\lambda, H_i] = 0 \tag{2.13}$$

and the commutativity

$$[\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}] = 0 \tag{2.14}$$

reproduce the respective properties (2.4) and (2.5) of B_λ. The most interesting property of $\hat{Q}_\lambda$, however, is that its eigenvalues $\phi(\lambda)$ on the joint eigenvectors Ψ_ν of H_i and Q_λ labelled with the quantum numbers ν

$$Q_\lambda \Psi_\nu = \phi_\nu(\lambda)\Psi_\nu \tag{2.15}$$

satisfy the *separation equation*, which is a certain differential or difference equation

$$\hat{W}\left(\lambda, -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}\lambda}; \{h_i\}\right)\phi_\nu(\lambda) = 0 \tag{2.16}$$

containing the eigenvalues h_i of H_i. In the classical limit the equation (2.16) goes over into the spectrality equation (2.9).

An important application of the spectrality property of BTs is that to the problem of *separation of variables* [7, 8]. Again, it is instructive to start with the quantum case. A separating operator $\hat{K}$ is, by definition, an operator, transforming the joint eigenfunctions Ψ_ν of H_i into the product

$$\hat{K}\Psi_\nu = c_\nu \prod_{i=1}^{n} \phi_\nu(\lambda_i) \tag{2.17}$$

of *separated functions* $\phi_\nu(\lambda)$ of one variable λ satisfying the separation equation (2.16). Since the coefficients c_ν in (2.17) can be chosen arbitrarily, abstractly speaking, there exist infinitely many separating operators $\hat{K}$. The difficult problem, however, is to find the ones which can be described as integral operators with explicitly given kernels.

Knowing a Q-operator gives one an immediate opportunity to construct plenty of separating operators. Indeed, consider the operator product $\hat{Q}_{\lambda_1\ldots\lambda_n} \equiv \hat{Q}_{\lambda_1}\ldots\hat{Q}_{\lambda_n}$ having the kernel $Q_{\lambda_1\ldots\lambda_n}(y; x)$ and for any function $\rho(y)$ introduce the operator

$$\hat{K}_\rho : \Psi(x) \to \int \mathrm{d}x \int \mathrm{d}y\, \rho(y) Q_{\lambda_1\ldots\lambda_n}(y; x)\Psi(x). \tag{2.18}$$

It is obvious from (2.15) that $\hat{K}_\rho$ is a separating operator, the coefficients c_ν being

$$c_\nu = \int \mathrm{d}y\, \rho(y)\Psi_\nu(y). \tag{2.19}$$

Since the eigenfunctions $\Psi_\nu(y)$ form a basis in the corresponding Hilbert space, the formula (2.19) provides a one-to-one correspondence between reasonably chosen classes of c_ν and $\rho(y)$. Therefore, arguably, the formula (2.18) describes all possible separating operators. Their kernels $K_\rho(\lambda; x)$ are given explicitly as multiple integrals

$$K_\rho(\lambda; x) = \int \mathrm{d}y \int \mathrm{d}\xi^{(1)} \ldots \int \mathrm{d}\xi^{(n-1)} \times \rho(y) Q_{\lambda_1}(y; \xi^{(1)}) Q_{\lambda_2}(\xi^{(1)}; \xi^{(2)}) \ldots Q_{\lambda_n}(\xi^{(n-1)}; x). \tag{2.20}$$

It is a straightforward task to present the classical analogue of the above argument. Consider the composition $B_{\lambda_1\ldots\lambda_n} = B_{\lambda_1} \circ \ldots \circ B_{\lambda_n}$ of Bäcklund transformations and the corresponding generating function $F_{\lambda_1\ldots\lambda_n}(y; x)$. Let us switch now the roles of y's and λ's treating λ's as dynamical variables and y's as parameters. Then $F_{\lambda_1\ldots\lambda_n}(y; x)$ becomes the generating function of the n-parametric canonical transformation K_y from (X, x) to (μ, λ) given by the equations

$$X_i = \frac{\partial F_{\lambda_1\ldots\lambda_n}}{\partial x_i} \qquad \mu_i = -\frac{\partial F_{\lambda_1\ldots\lambda_n}}{\partial \lambda_i}. \tag{2.21}$$

It follows directly from (2.9) that the pairs (λ_i, μ_i) satisfy the separation equations

$$W(\lambda_i, f(\mu_i); \{H_j\}) = 0 \tag{2.22}$$

which constitutes exactly the definition of the separating canonical transformation in the classical case [7].

The above construction corresponds in the quantum case to setting $\rho(y) = \delta(y_1 - \bar{y}_1)\dots\delta(y_n - \bar{y}_n)$ where $\bar{y}_i$ are some constants. It remains an open question what could be the classical analogue of the formula (2.18) for generic $\rho(y)$.

As the last general remark before passing to the examples, we would like to stress that for the finite-dimensional systems the composition of n BTs with n being the number of degrees of freedom is a sort of 'universal' BT in the sense that any other canonical transformation preserving the Hamiltonians H_i must be expressible in terms of $B_{\lambda_1\dots\lambda_n}$. To observe it one can again use the fact that in the angle coordinate B_λ acts as a shift $\varphi_i \to \varphi_i + b_i(\lambda)$. For generic $b_i(\lambda)$ the sum $b_i(\lambda_1) + \cdots + b_i(\lambda_n)$ must then cover the n-dimensional Liouville torus which results in the universality of $B_{\lambda_1\dots\lambda_n}$.

3. Periodic Toda lattice

Our first example is the periodic Toda lattice [12, 13] for which there exist two alternative Lax operators associated, as we shall show, with two different BTs. The standard and quite well studied BT [1, 10, 12, 14] which we denote here as B_λ is associated, in the sense defined in the previous section, to the 2×2 Lax matrix (or, monodromy matrix [15]) $L(u; X, x)$ defined as the product of local L-operators

$$L(u) = \ell_n(u)\dots\ell_2(u)\ell_1(u) \tag{3.1}$$

$$\ell_i(u) \equiv \ell_i(u; X_i, x_i) = \begin{pmatrix} u + X_i & -\mathrm{e}^{x_i} \\ \mathrm{e}^{-x_i} & 0 \end{pmatrix}. \tag{3.2}$$

The characteristic polynomial of $L(u)$ is quadratic in v

$$W(u, v) \equiv \det(v - L(u)) = v^2 - t(u)v + 1 \tag{3.3}$$

and the commuting Hamiltonians H_i are obtained from the expansion of the only non-trivial spectral invariant $t(u) \equiv \operatorname{tr} L(u)$

$$t(u) = u^n + H_1 u^{n-1} + \cdots + H_n. \tag{3.4}$$

In particular,

$$\frac{1}{2}H_1^2 - H_2 = \sum_{i=1}^{n} \left(\frac{1}{2}X_i^2 + \mathrm{e}^{x_{i+1} - x_i} \right) \tag{3.5}$$

is the standard periodic Toda Hamiltonian (in this section we use the periodicity convention $i + n \equiv i$ for the indices i).

The Bäcklund transformation B_λ is obtained from the generating function

$$F_\lambda(y; x) = \sum_{i=1}^{n} (\mathrm{e}^{x_i - y_i} - \mathrm{e}^{y_{i+1} - x_i} - \lambda(x_i - y_i)) \tag{3.6}$$

and, according to (2.3), is implicitly described by the equations

$$X_i = \mathrm{e}^{x_i - y_i} + \mathrm{e}^{y_{i+1} - x_i} - \lambda \qquad Y_i = \mathrm{e}^{x_i - y_i} + \mathrm{e}^{y_i - x_{i-1}} - \lambda. \tag{3.7}$$

The characteristic properties of the BT are verified easily. The *invariance of the Hamiltonians* can be established using the equality [10]

$$M_{i+1}(u, \lambda)\ell_i(u; X_i, x_i) = \ell_i(u; Y_i, y_i)M_i(u, \lambda) \tag{3.8}$$

where

$$M_i(u, \lambda) \equiv M_i(u, \lambda; x_{i-1}, y_i) = \begin{pmatrix} 1 & -\mathrm{e}^{y_i} \\ \mathrm{e}^{-x_{i-1}} & \lambda - u - \mathrm{e}^{y_i - x_{i-1}} \end{pmatrix} \tag{3.9}$$

which one can verify directly using equations (3.7). Due to the periodic boundary conditions, the local gauge transformation (3.8) results in the spectrum-preserving similarity transformation

$$M_1(u,\lambda)L(u;X,x) = L(u;Y,y)M_1(u,\lambda) \tag{3.10}$$

for $L(u)$ which proves the invariance (2.4) of the Hamiltonians.

The direct proof of the *commutativity* (2.5) of the BTs can be found in [1, 12, 14].

To prove the *spectrality* equality (2.9) which in this case takes the form $\det(\mathrm{e}^{-\mu} - L(\lambda)) = 0$ we shall apply a modified version of the argument used in [2, 10] for the quantum case. Note, first, that in our case

$$\mu = -\frac{\partial F_\lambda}{\partial \lambda} = \sum_{i=1}^{n}(x_i - y_i) \tag{3.11}$$

as follows from (3.6) and (2.8). It suffices then to show that $\mathrm{e}^{-\mu}$ is an eigenvalue of the matrix $L(\lambda)$. We shall construct explicitly the corresponding eigenvector ω_1

$$L(\lambda;X,x)\omega_1 = \mathrm{e}^{-\mu}\omega_1. \tag{3.12}$$

From (3.9) it follows that $\det(M_i(u,\lambda)) = \lambda - u$. It is easy to see that for $u = \lambda$ the matrix $M_i(\lambda,\lambda)$ has the unique, up to a scalar factor, null-vector

$$\omega_i = \begin{pmatrix} \mathrm{e}^{y_i} \\ 1 \end{pmatrix} \qquad M_i(\lambda,\lambda)\omega_i = 0. \tag{3.13}$$

Using the identity (3.10) we conclude that

$$M_1(\lambda,\lambda)L(\lambda;X,x)\omega_1 = 0 \tag{3.14}$$

which, combined with the uniqueness of the null-vector ω_1 of M_1, implies that ω_1 is an eigenvector of $L(\lambda;X,x)$. To determine the corresponding eigenvalue, we apply the same argument to the identity (3.8) obtaining the equality $M_{i+1}(\lambda;\lambda)\ell_i(\lambda;X_i,x_i)\omega_i = 0$ from which it follows that $\ell_i(\lambda;X_i,x_i)\omega_i \sim \omega_{i+1}$. The direct calculation shows that

$$\ell_i(\lambda;X_i,x_i)\omega_i = \mathrm{e}^{y_i - x_i}\omega_{i+1}. \tag{3.15}$$

It only remains to use the formulae (3.1) and (3.11) to arrive finally at (3.12). Actually, we could skip the discussion of null-vectors of M_i and derive (3.12) directly from (3.15). In more complicated situations, however, it may be easier to find ω as the null-vector of M and then to determine the corresponding eigenvalue of $L(\lambda)$.

Note that the vectors ω_i are the classical counterparts of Baxter's [11] vacuum vectors.

Let us now examine the alternative Lax operator [12, 13] given by the $n \times n$ matrix $\mathcal{L}(v;X,x)$ with the components

$$\mathcal{L}_{jk}(v;X,x) = -X_j\delta_{jk} + v^{-1/n}\,\mathrm{e}^{x_j - x_k}\delta_{j,k+1} + v^{1/n}\delta_{j+1,k}. \tag{3.16}$$

The duality between the Lax operators $\mathcal{L}(v)$ and $L(u)$ is expressed in the switching of the roles of the parameters u and v. The characteristic polynomial $\mathcal{W}(v,u) \equiv \det(u - \mathcal{L}(v))$ of the Lax operator (3.16) produces the same Hamiltonians H_i and the same spectral curve as $W(u,v)$, as follows from the identity

$$\det(v - L(u)) = -v\det(u - \mathcal{L}(v)). \tag{3.17}$$

For other examples of similar duality, see [16].

The swapping of u and v corresponds to switching the roles of the parameters λ and μ in the BT. For the new Bäcklund transformation $\mathcal{B}_\mu$ associated with the Lax operator $\mathcal{L}(v)$ the formulae (3.6), (3.7) and (3.11) remain the same but their interpretation changes. The BT is parametrized now by the parameter μ which becomes a numerical constant. The equality (3.11) is now reinterpreted as a constraint on the variables x_i and y_i. The parameter λ is reinterpreted, respectively, as the Lagrange multiplier for the constraint (3.11) and becomes a dynamical variable which can be defined from the equations (3.7).

The characteristic properties of BTs are verified for $\mathcal{B}_\mu$ in very much the same manner as for B_λ. The *invariance of the Hamiltonians* follows from the invariance of the spectrum of $\mathcal{L}(v)$ which, in turn, follows from the easily verified identity

$$\mathcal{M}(v)\mathcal{L}(v; X, x) = \mathcal{L}(v; Y, y)\mathcal{M}(v) \tag{3.18}$$

with the matrix $\mathcal{M}(v) \equiv \mathcal{M}(v; x, y)$ given by its components

$$\mathcal{M}_{jk}(v) = -\delta_{jk} + v^{-1/n}\, \mathrm{e}^{y_j - x_k}\delta_{j,k+1}. \tag{3.19}$$

The *commutativity*, as shown in section 2 follows from the canonicity and the invariance.

To prove the *spectrality* equality $\det(\lambda - \mathcal{L}(\mathrm{e}^{-\mu})) = 0$, it suffices, similarly to the case of the Lax operator $L(u)$, to present the eigenvector Ω of the matrix $\mathcal{L}(\mathrm{e}^{-\mu})$ corresponding to the eigenvalue λ

$$\mathcal{L}(\mathrm{e}^{-\mu})\Omega = \lambda\Omega. \tag{3.20}$$

Again, Ω can be determined as the null-vector of $\mathcal{M}(\mathrm{e}^{-\mu})$

$$\mathcal{M}(\mathrm{e}^{-\mu})\Omega = 0. \tag{3.21}$$

Note that the uniqueness of Ω follows from the easily verified identity $\det\big(z - \mathcal{M}(v)\big) = (z+1)^n - v^{-1}\,\mathrm{e}^{-\mu}$ which implies that the spectrum of $\mathcal{M}(\mathrm{e}^{-\mu})$ consists of n non-degenerate eigenvalues, 0 being one of them. From (3.21) one easily derives the recurrence relation for the components of Ω

$$\Omega_j = \Omega_{j-1} \exp\left(y_j - x_{j-1} + \frac{\mu}{n}\right) \tag{3.22}$$

which determines Ω up to a constant factor. It remains to verify the identity (3.20) which can be done by a direct calculation using the expressions (3.16) for the matrix $\mathcal{L}(v)$, (3.11) for μ and (3.7) for X_i.

4. Elliptic Ruijsenaars model

Our second example is the relativistic generalization due to Ruijsenaars [17] of the elliptic Calogero–Moser [18] many-body system. For the non-relativistic Calogero–Moser system a BT was found in [19]. In [5] a discrete-time dynamics was constructed for the elliptic Ruijsenaars model. As we show later, the discrete-time evolution transformation found in [5] has all the properties of a BT if the parameter p in [5] is specified in a proper way.

We use here the notation of [5] with a few exceptions: our parameter ξ equals $-\lambda$ from [5], and our $\mathrm{e}^{-\lambda}$ corresponds to p from [5]. As in the case of the Toda lattice, there exist two dual BTs: B_λ and $\mathcal{B}_\mu$. The standard Lax operator for the Ruijsenaars model, as shown later, is associated with $\mathcal{B}_\mu$. Since the dual Lax operator is so far unknown, we describe here only the transformation $\mathcal{B}_\mu$.

Following [5, 8], we introduce the Lax operator $\mathcal{L}(v; X, x)$ for the n-particle (A_{n-1} type) Ruijsenaars system as the $n \times n$ matrix with the entries

$$\mathcal{L}_{ij}(v) = -\mathrm{e}^{X_i}\frac{\sigma(\xi)\sigma(v + x_i - x_j - \xi)}{\sigma(v)\sigma(x_i - x_j - \xi)}\prod_{k\neq i}\frac{\sigma(x_i - x_k + \xi)}{\sigma(x_i - x_k)} \tag{4.1}$$

where $\sigma(x)$ is the Weierstrass sigma function and ξ is a constant.

The commuting Hamiltonians

$$H_i = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=i}} \exp\left(\sum_{j\in J} X_j\right) \prod_{\substack{j\in J\\ k\in\{1,\ldots,n\}\setminus J}} \frac{\sigma(x_j - x_k + \xi)}{\sigma(x_j - x_k)} \qquad i = 1, \ldots, n \tag{4.2}$$

are generated from the characteristic polynomial of the matrix $\mathcal{L}(v)$ (4.1)

$$\det(\mathcal{L}(v) - u) = \sum_{j=0}^{n} (-u)^{n-j} H_j \frac{\sigma(v - j\xi)}{\sigma(v)} \tag{4.3}$$

where we assume $H_0 \equiv 1$.

The Bäcklund transformation $\mathcal{B}_\mu$ is given by the equations

$$e^{X_i} = e^{-\lambda} \prod_{j\neq i} \frac{\sigma(x_i - x_j - \xi)}{\sigma(x_i - x_j + \xi)} \prod_{k=1}^{n} \frac{\sigma(x_i - y_k + \xi)}{\sigma(x_i - y_k)} \tag{4.4}$$

$$e^{Y_i} = e^{-\lambda} \prod_{k=1}^{n} \frac{\sigma(x_k - y_i + \xi)}{\sigma(x_k - y_i)} \tag{4.5}$$

where λ is considered as the Lagrange multiplier corresponding to the constraint

$$\mu = n\xi + \sum_{k=1}^{n} (x_k - y_k). \tag{4.6}$$

Note here that the variable λ in formulae (4.4) and (4.5), describing the dicrete-time dynamics, appeared as p in [5], but the conjugated variable μ did not. Notice also that λ was treated in [5] as an extra parameter, not as a Lagrange multiplier corresponding to a constraint.

The generating function of the canonical transformation $\mathcal{B}_\mu$ is expressed in terms of the function

$$\mathcal{S}(x) = \int^{x} \ln\sigma(y)\,dy \tag{4.7}$$

as follows

$$\begin{aligned}\mathcal{F}_\lambda(y; x) = -\lambda \sum_{i=1}^{n} (x_i - y_i + \xi) + \sum_{i<j} (\mathcal{S}(x_i - x_j - \xi) - \mathcal{S}(x_i - x_j + \xi)) \\ + \sum_{i,j=1}^{n} (\mathcal{S}(x_i - y_j + \xi) - \mathcal{S}(x_i - y_j)).\end{aligned} \tag{4.8}$$

The verification of the characteristic properties of the BT for $\mathcal{B}_\mu$ proceeds in the same way as in the case of the Toda lattice.

The *invariance of the Hamiltonians H_i* follows from the identity (see [5] for the proof)

$$\mathcal{M}(v)\mathcal{L}(v; X, x) = \mathcal{L}(v; Y, y)\mathcal{M}(v) \tag{4.9}$$

where the matrix $\mathcal{M}(v) \equiv \mathcal{M}(v; x, y)$ is defined as

$$\mathcal{M}_{ij}(v) = \frac{\sigma(v + y_i - x_j - \xi)}{\sigma(y_i - x_j - \xi)} \prod_{k\neq i} \frac{\sigma(y_i - y_k + \xi)}{\sigma(y_i - y_k)} \prod_{k} \frac{\sigma(x_k - y_i + \xi)}{\sigma(x_k - y_i)}. \tag{4.10}$$

The *commutativity*, as usual, is a consequence of canonicity and invariance (see section 2).

To prove the *spectrality* equality which takes the form $\det(e^{-\lambda} - \mathcal{L}(\mu)) = 0$ it is sufficient, like in the case of the Toda lattice, to find the eigenvector Ω of the matrix $\mathcal{L}(\mu)$

corresponding to the eigenvalue $e^{-\lambda}$. Let us show that, up to a constant multiplier, the components of the eigenvector Ω are

$$\Omega_i = \frac{\prod_{k=1}^{n} \sigma(x_i - y_k + \xi)}{\prod_{k\neq i} \sigma(x_i - x_k)}. \tag{4.11}$$

The equality

$$\mathcal{L}(\mu)\Omega = e^{-\lambda}\Omega \tag{4.12}$$

or

$$\sum_{j=1}^{n} \mathcal{L}_{ij}(\mu)\Omega_j = e^{-\lambda}\Omega_i \tag{4.13}$$

after the substitutions (4.1) for $\mathcal{L}_{ij}$, (4.6) for μ and (4.4) for e^{X_i} is reduced to the following identity for sigma functions

$$\sum_{j=1}^{n} \sigma(\mu + x_i - x_j - \xi) \prod_{k=1}^{n} \sigma(x_j - y_k + \xi) \prod_{k\neq j} \frac{\sigma(x_i - x_k - \xi)}{\sigma(x_j - x_k)} = \sigma(\mu) \prod_{k=1}^{n} \sigma(x_i - y_k). \tag{4.14}$$

Due to the symmetry, it is sufficient to prove (4.14) only for $i = 1$. Let $i = 1$ and $n \geqslant 2$. Consider both sides of the equality (4.14) as functions of x_n. It is easy to see that they are holomorphic in x_n (the apparent poles in the left-hand side being cancelled) and have the same quasi-periodicity properties. From the basic properties of sigma functions [20] it follows that it is sufficient to verify the equality of left-hand side and right-hand side only in one arbitrary point $x_n = \bar{x}$ with the only condition $\bar{x} \neq \mu - x_n$. Choosing $\bar{x} = y_n - \xi$ we observe that (4.14) is reduced to the similar identity of order $n - 1$. The proof follows then by induction in n since the case when $n = 1$ is trivial.

As in section 3, the vector Ω (4.11) is again the null-vector of the matrix $\mathcal{M}(\mu)$, i.e.

$$\mathcal{M}(\mu)\Omega = 0. \tag{4.15}$$

The corresponding identity for sigma functions

$$\sum_{j=1}^{n} \sigma(\mu + y_i - x_j - \xi) \frac{\prod_{k\neq i} \sigma(x_j - y_k + \xi)}{\prod_{k\neq j} \sigma(x_j - x_k)} = 0 \tag{4.16}$$

follows from the identity

$$\sum_{j=1}^{n} \frac{\prod_{k=1}^{n} \sigma(x_j - z_k)}{\prod_{k\neq j} \sigma(x_j - x_k)} = 0 \qquad \text{if} \qquad \sum_{k=1}^{n} (z_k - x_k) = 0 \tag{4.17}$$

(cf [20], p 451) when one substitutes $z_k = y_k - \xi$ for $k \neq i$ and $z_i = \mu + y_i - \xi$.

5. Discussion

We have studied three new aspects of Bäcklund transformations. These are spectrality, dual BTs and the application of BTs to the problem of separation of variables. As demonstrated in section 2, the composition of n BTs, being a 'universal' (n-parametric) BT, provides a separation of variables which has n arbitrary parameters, and thereby defines a 'universal' (n-parametric) family of separating transformations. The connection between the 'universal' BT and the 'universal' SoV is intriguing and has yet to be studied in detail.

Although we have discussed in the present paper the classical case only, our primary motivation comes from the quantum case. The main problem in the quantum case is to construct Baxter's Q-operator which is a quantum analogue of the Bäcklund transformation. For the trigonometric case of the Ruijsenaars system, i.e. for the case of multivariable (A_{n-1}-type) Macdonald polynomials, we have succeeded in describing explicitly such a quantum analogue of the transformation $\mathcal{B}_\mu$ introduced in section 4. These results will be reported elsewhere.

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